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THE COMPLETE QUADRILATERAL.

BY JOHN WENTWORTH CLAWSON.

Introduction. It is the purpose of this paper to discuss systematically the principal points and lines related to the complete quadrilateral, referring, as far as practicable, to the discoverers of theorems which are not new.* In part I more than thirty theorems are given of which (7) and (31) are probably new, the others having been collected from various sources; in part II sixteen new theorems are derived by an inversion; in part III some properties of the cyclic quadrangle are collected, from which, in part IV, by a polar reciprocation, fourteen other new properties of the complete quadrilateral are deduced. The methods of proof used are those of elementary pure geometry.

Throughout the paper ABC will be taken to denote the angle through which the line AB, taken as a whole, must be rotated in order for it to coincide with BC, taken as a whole, the rotation taking place in the positive direction. With this understanding, two angles are regarded as equivalent if they differ only by multiples of π; so $n\pi = 0$. The symbol $=$ is used throughout the paper in this sense.†

As it is impossible to print figures to illustrate all the theorems of this paper, the reader is urged to draw his own figures wherever it is necessary.

PART I. POINTS AND LINES RELATED TO THE COMPLETE QUADRILATERAL.

1. **The focal point.** The complete quadrilateral consists of four coplanar straight lines, l_1, l_2, l_3, l_4, intersecting in three pairs of opposite vertices, A_{14}, A_{23}; A_{13}, A_{24}; A_{12}, A_{34}. The lines l_2, l_3, l_4, determine a triangle. Let the center of the circle, $\mathcal{C}_1$, which circumscribes this triangle be C_1. Three other such triangles and circumscribed circles exist.

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* In 1828 the famous geometer Steiner proposed for solution (Gergonne's Annales, vol. 18, p. 302) a list of ten theorems relating to the complete quadrilateral. No other notable attempt to collect the properties of this interesting figure appears to have been made until 1901, when Léon Ripert published a paper (Compte Rendu de l'Association francaise pour l'Avancement des Sciences, vol. 30, part 2, p. 91), in which he derived a number of theorems, using the method of trilinear coördinates. Other theorems are scattered through the journals mentioned in the foot notes and elsewhere. Theorems in elementary geometry have appeared in so many places that it is impossible to hope that I have traced all the theorems given to their sources, or that I have not missed altogether some important properties of the quadrilateral.

† R. A. Johnson pointed out in the American Mathematical Monthly, vol. 24 (1917), p. 101, that proofs in elementary geometry may be made to cover all possible figures if the above conventions are adopted.

(1) The four circumscribed circles are concurrent at a point.* This will be called the *focal point* of the quadrilateral, and will be denoted by F.

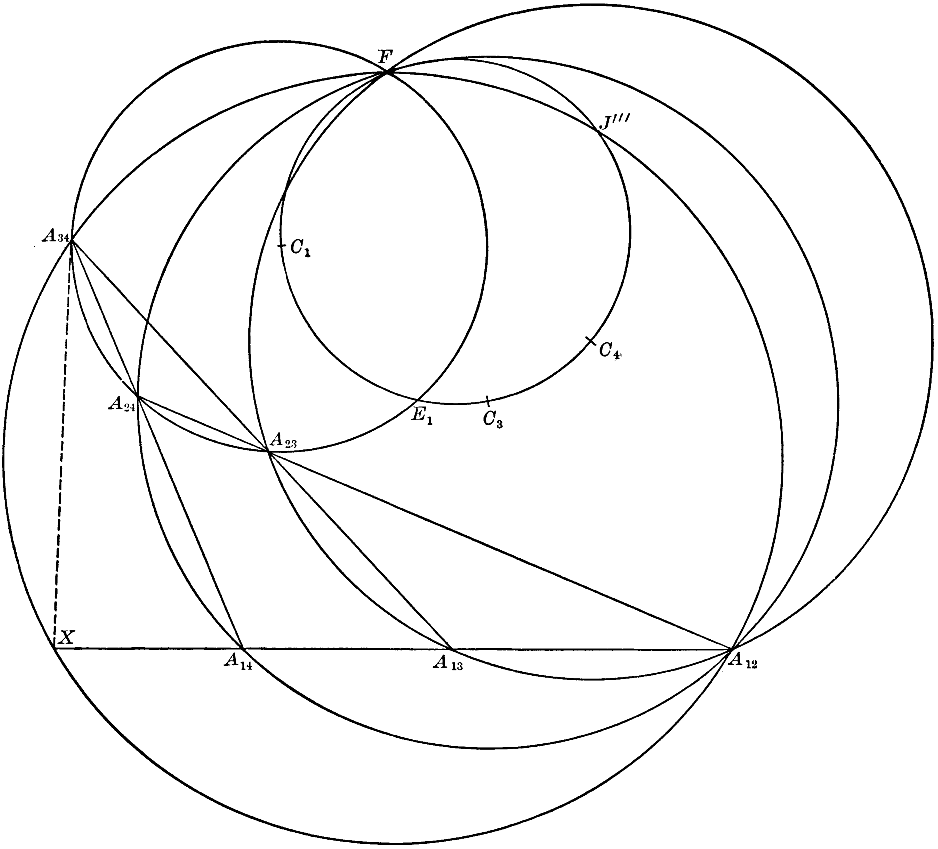

FIG. 1.

For, if F is the intersection of $\mathcal{C}_1$ and $\mathcal{C}_3$ (Fig. 1), considering points on $\mathcal{C}_1$, $A_{24}FA_{23} = A_{24}A_{34}A_{23}$; and, considering points on $\mathcal{C}_3$, $A_{12}FA_{24} = A_{12}A_{14}A_{24}$; by addition of equals, $A_{12}FA_{23} = A_{12}A_{13}A_{23}$; and therefore the circle $\mathcal{C}_4$ also passes through F. Similarly it may be proved that the circle $\mathcal{C}_2$ passes through the focal point.

F has been called the *Wallace point* and the *Miquel point* of the quadrilateral. It seems wiser, however, to avoid the use of a proper name if a short descriptive name can be coined.

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* This theorem was first stated by "Scoticus" in 1804 in Leybourn's Mathematical Repository, vol. 1, part 1, p. 170. Mackay, in the Proceedings of the Edinburgh Mathematical Society, vol. 9 (1890), pp. 83–91, says that "Scoticus" is a pseudonym of Dr. William Wallace. It is the first of Steiner's theorems, l. c.; and was proved again by Miquel in Liouville's Journal de Mathématiques, vol. 3 (1838), pp. 485–487. The proof above is that of the Repository modified to suit Johnson's notation.

(2) The four points where the tangent at A_{12} to $\mathcal{C}_3$ cuts l_3, where the tangent at A_{12} to $\mathcal{C}_4$ cuts l_4, where the tangent at A_{34} to $\mathcal{C}_1$ cuts l_1, where the tangent at A_{34} to $\mathcal{C}_2$ cuts l_2, are concyclic with A_{12} and A_{34}. This circle and two circles similarly obtained are concurrent at F.*

For, let the tangent at A_{34} to $\mathcal{C}_1$ meet l_1 at X. Then,

$$A_{13}XA_{34} + XA_{34}A_{13} + A_{34}A_{13}X = 0;$$

hence, since $A_{34}X$ is a tangent to $\mathcal{C}_1$, $A_{12}XA_{34} + A_{34}A_{24}A_{23} + A_{34}A_{13}A_{12} = 0$. Again, $A_{12}FA_{34} + FA_{34}A_{24} + A_{34}A_{24}A_{12} + A_{24}A_{12}F = 0$. But, considering points on $\mathcal{C}_2$, $FA_{34}A_{24} = FA_{13}A_{12}$; and, considering points on $\mathcal{C}_4$, $A_{24}A_{12}F = A_{34}A_{13}F$; hence, $A_{12}FA_{34} + FA_{13}A_{12} + A_{34}A_{24}A_{12} + A_{34}A_{13}F = 0$; and therefore $A_{12}FA_{34} + A_{34}A_{24}A_{23} + A_{34}A_{13}A_{12} = 0$. Comparing this with the second equation, we see that $A_{12}XA_{34} = A_{12}FA_{34}$ and hence that A_{12}, X, A_{34}, F are concyclic. Similarly the other facts may be proved.

Draw the three diagonals, n' joining A_{14} to A_{23}, n'' joining A_{13} to A_{24}, n''' joining A_{12} to A_{34}; let n', n'' intersect at D_{12}, this and two points similarly obtained forming the diagonal triangle $D_{12}D_{13}D_{23}$.

Let F_1', F_2', F_1'', F_2'', F_1''', F_2''' be the focal points of the quadrilaterals $n''n'''l_2l_3$, $n''n'''l_1l_4$, $n'n'''l_1l_3$, $n'n'''l_2l_4$, $n'n''l_1l_2$, $n'n''l_3l_4$ respectively. Call the centers of the circles circumscribing l_2l_3n'', l_2l_3n''', C_{23}'', C_{23}''' respectively. Then $C_1C_{23}''C_4C_{23}'''$ is a parallelogram, since C_1C_{23}'' and C_4C_{23}''' are both perpendicular to l_2, and C_1C_{23}''' and C_4C_{23}'' are both perpendicular to l_3. Again, since FA_{23} and $F_1'A_{23}$ are the common chords of circles $\mathcal{C}_1$, $\mathcal{C}_4$ and $\mathcal{C}_{23}''$, $\mathcal{C}_{23}'''$ respectively, the point of intersection of the diagonals of this parallelogram is equally distant from A_{23}, F and F_1'. It is further easily proved that a circle can be drawn with this point as center, passing through A_{23}, F, F_1' and the middle points of $A_{13}A_{34}$ and $A_{12}A_{24}$. In future the notation (AB) will be used for the middle point of the line joining A to B. Hence

(3) The circles determined by A_{23}, $(A_{12}A_{24})$, $(A_{13}A_{34})$; by A_{14}, $(A_{12}A_{13})$, $(A_{34}A_{24})$; and four other such sets of points pass respectively through F_1', F_2', F_1'', F_2'', F_1''', F_2''' and all six circles pass through F.

In exactly the same way it can be proved that

(3a) The circle determined by D_{23}, $(A_{13}A_{24})$ and $(A_{12}A_{34})$ passes through F_1' and F_2'†; and that two other such circles can be drawn.

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* This theorem is given as an unproved exercise in Casey's Analytic Geometry, second edition, 1893, Ex. 81, page 535. The proof here given is new.

† The theorem (3a) was set as Question 252 in Journal de Mathématiques élémentaires et spéciales and solved by Rivard in the same Journal, vol. 5 (1881), p. 118. I have expanded this theorem into the group above.

If we anticipate the result of (8), that the middle points of the diagonals are collinear, it is clear that the three circles of (3*a*) are three of the circles circumscribed to triangles of the quadrilateral whose six vertices are D_{12}, D_{13}, D_{23} and the middle points of the three diagonals. Therefore

(3*b*) The circles $F_1'F_2'D_{23}$, $F_1''F_2''D_{13}$ and $F_1'''F_2'''D_{12}$ are concurrent, and the circle $D_{12}D_{13}D_{23}$ passes through the same point.

It may also be proved that

(3*c*) The circles $FF_1'A_{23}$, $FF_2'A_{14}$ and $F_1'F_2'D_{23}$ are concurrent; and that there are two other sets of three concurrent circles, one from the set (3*a*) and two from the set (3).

2. **The circumcentric circle.** We shall prove the theorem:

(4) The points C_1, C_2, C_3, C_4 and F are concyclic.*

The circle, $\mathcal{C}$, on which they lie will be called the *circumcentric circle* of the quadrilateral.

For, $FC_3C_1 = FA_{12}A_{24}$ and $FC_4C_1 = FA_{12}A_{23}$, since a central angle is double an angle at the circumference subtended by the same arc. Therefore F, C_3, C_1 and C_4 are concyclic; similarly it can be shown that F, C_2, C_1 and C_3 are concyclic.

(5) The lines C_2A_{34}, C_3A_{24}, C_4A_{23} are concurrent at a point common to $\mathcal{C}_1$ and $\mathcal{C}$.†

Call this point E_1. Three other points are similarly found.

For, let C_2A_{34} and C_3A_{24} meet at E_1. Then

$$E_1A_{24}A_{34} + A_{24}A_{34}E_1 + A_{34}E_1A_{24} = 0.$$

But

$$E_1A_{24}A_{34} = (\pi/2) - \tfrac{1}{2}A_{14}C_3A_{24} = (\pi/2) - A_{14}A_{12}A_{24};$$

and

$$A_{24}A_{34}E_1 = (\pi/2) - \tfrac{1}{2}A_{34}C_2A_{14} = (\pi/2) - A_{34}A_{13}A_{14}.$$

Adding and substituting in the first equation, we get

$$A_{34}E_1A_{24} = A_{34}A_{13}A_{14} + A_{14}A_{12}A_{24} = A_{34}A_{23}A_{24}.$$

Hence E_1 lies on the circle $\mathcal{C}_1$.

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* This is the second of Steiner's theorems, l. c. It was restated by T. S. Davies in Leybourn's Mathematical Repository, vol. 6 (1835), Question 555, and proved there by him. Probably Davies discovered this theorem independently of Steiner; that he had been studying the quadrilateral for some time is evidenced by his statement in connection with the solution of Question 524 in the same volume of the Repository and by J. S. Mackay's remark in the Proceedings of the Edinburgh Mathematical Society, l. c., that Davies had proposed a question in 1821 in the Leeds Correspondent dealing with a property of this figure. After proving the theorem above, Davies proceeds with several other properties including a special case of (5). This circle is called the *center circle* by Gallatly in his Modern Geometry of the Triangle, London, N. D., page 5. It was called the *eight point circle* by Hermes after he found the four new points of (5) on it. The above is essentially Davies' proof.

† This theorem was partially stated and proved by Davies, l. c. The complete statement and proof, substantially equivalent to the above, is due to Hermes, Nouvelles Annales, vol. 18 (1859), page 359.

Again $C_2C_4C_3 = A_{13}FA_{12}$, since C_2C_4 and C_3C_4 are respectively perpendicular to the common chords FA_{13} and FA_{12}; but, considering points on the circle $\mathcal{C}_4$, the latter equals $A_{13}A_{23}A_{12}$ or $A_{34}A_{23}A_{24}$. But $A_{34}A_{23}A_{24}$ was proved above to be equal to $C_2E_1C_3$. Hence E_1 lies on the circle $\mathcal{C}$.

It follows at once that

(5*a*) Tangents at A_{34} to $\mathcal{C}_2$, at A_{23} to $\mathcal{C}_4$, at A_{24} to $\mathcal{C}_3$, meet at a point diametrically opposite to E_1 in $\mathcal{C}_1$;[*] also that

(5*b*) The pedal line of E_1 with respect to the triangle $l_2l_3l_4$ is parallel to l_1.[†]

(6) The perpendicular bisectors of $A_{13}A_{14}$ and $A_{23}A_{24}$ intersect at a point on the circumcentric circle.[‡]

This point will be denoted by T_{34}. There are six such points.

For, $C_1T_{34}C_2 = A_{24}A_{12}A_{14}$, the sides being respectively perpendicular; the latter is equivalent to $A_{24}FA_{14}$, the four points lying on $\mathcal{C}_3$;

$$A_{24}FA_{14} = C_1C_3C_2,$$

since the line of centers of the circles $\mathcal{C}_1$, $\mathcal{C}_3$ is perpendicular to the common chord $A_{24}F$ and C_2C_3 similarly perpendicular to $A_{14}F$. Then

$$C_1T_{34}C_2 = C_1C_3C_2,$$

and the four points are concyclic.

T_{34} is evidently the point diametrically opposite to A_{12} in the fifth of the circles of (3).

The six vertices of the quadrilateral taken in threes determine sixteen circles. Four of these are $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$. Denote by C_{13}', C_{14}'', C_{34}''', $\cdots$ the centers of the circles circumscribing the triangles l_1l_3n', l_1l_4n'', l_3l_4n''', $\cdots$.

It is evident that C_1, C_{24}''' and C_{34}'' are collinear, since the three circles are coaxal. Again C_{12}', C_{13}', C_{24}' and C_{34}' are collinear, for the same reason. It is easily seen then that the sixteen centers lie three by three on twelve lines, the perpendicular bisectors of pairs of vertices on the same l, and four by four on three lines, the perpendicular bisectors of the diagonals.

Now $A_{24}A_{34}C_1 = (\pi/2) - A_{34}A_{23}A_{24} = A_{12}A_{34}C_{23}'''$. Therefore

$$A_{24}A_{34}A_{12} = C_1A_{34}C_{23}''',$$

on adding $C_1A_{34}A_{12}$. And $C_1C_{24}'''C_{23}''' = A_{24}A_{34}A_{12}$, since C_1C_{24}''', $C_{24}'''C_{23}'''$ are perpendicular respectively to l_4, n'''. Therefore

$$C_1C_{24}'''C_{23}''' = C_1A_{34}C_{23}''';$$

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[*] Wetzig, Crelle's Journal, vol. 72 (1870), p. 351.

[†] Gallatly, Modern Geometry of the Triangle, p. 27.

[‡] This theorem is given by Ripert, l. c., who proves it analytically. I have furnished the above proof.

and C_1, C_{23}''', C_{24}''' and A_{34} are concyclic. Similarly C_2, C_{13}''', C_{14}''', A_{34}; C_3, C_{14}''', C_{24}''', A_{12}; C_4, C_{13}''', C_{23}''', A_{12} are concyclic.

Let the circles $C_1C_{24}'''A_{34}$ and $C_2C_{13}'''A_{34}$ intersect again at J'''. Now $C_1J'''A_{34} = C_1C_{24}'''A_{34} = A_{24}A_{12}A_{34}$; and

$$C_2J'''A_{34} = C_2C_{13}'''A_{34} = A_{13}A_{12}A_{34}.$$

Therefore $C_1J'''C_2 = A_{23}A_{12}A_{13} = A_{24}A_{12}A_{14} = C_1C_3C_2$, by (6). Consequently J''' is on the circumcentric circle. Similarly we prove that

(7) The four circles determined by the circumcenters of (*a*) the three triangles determined by A_{34} and two of the vertices on l_1, (*b*) the three triangles determined by A_{34} and two of the vertices on l_2, (*c*) the three triangles determined by A_{12} and two of the vertices on l_3, (*d*) the three triangles determined by A_{12} and two of the vertices on l_4 are concurrent at J''', a point on the circumcentric circle.*

Two other points are associated with the other pairs of opposite vertices.

Moreover, since $C_1J'''A_{34} = C_1C_{24}'''A_{34} = A_{24}A_{12}A_{34}$; and

$$C_1J'''F = C_1C_3F = A_{24}A_{12}F;$$

therefore $A_{34}J'''F = A_{34}A_{12}F$. Hence

(7*a*) The points A_{34}, A_{12}, F and J''' are concyclic; similarly A_{13}, A_{24}, F, J'' and A_{14}, A_{23}, F, J' are concyclic.*

These are the circles of (2).

3. The mid-diagonal line. We shall prove that

(8) The middle points of the diagonals of the quadrilateral are collinear on a line, m, which contains the centers of all conics touching the four sides of the quadrilateral.†

This line will be called the *mid-diagonal line* of the quadrilateral; Ripert calls it the *axis of mean distances;* it is sometimes called the *Newtonian*.

Let one of the inscribed conics touch l_1 at Z_1, l_4 at Z_4, l_3 at Z_3. Then, applying Brianchon's theorem to the hexagons $Z_4A_{24}A_{12}Z_1A_{13}A_{34}$, $Z_3A_{23}A_{12}Z_1A_{14}A_{34}$, we see that Z_4Z_1, Z_3Z_1 pass respectively through D_{23}, D_{13}. Let the lines joining A_{14}, A_{13} respectively to (Z_4Z_1), (Z_3Z_1) meet at X. Then X is the center of the inscribed conic.

Draw $A_{14}Y$, $A_{13}Y$ parallel respectively to Z_4Z_1, Z_3Z_1 intersecting at Y.

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* Theorems (7) and (7*a*) are new.

† This theorem is ascribed to Newton by Steiner, l. c., and others; it is sometimes ascribed to Gauss, for example by Schlömilch, Berichte der Gesellschaft der Wissenshaften zu Leipzig, vol. 9, 1854, p. 4, and by Hall, Messenger of Mathematics, vol. 4, 1868, p. 137. In fact the theorem is an immediate consequence of lemma XXV, cor. 3, sect. V, vol. 1 of the Principia, 1684. The first part of it was probably first explicitly stated by Connor in the Ladies' Diary for 1795. The proof above is due to Poncelet, Gergonne's Annales, vol. 12, p. 109. The mid-diagonal is thus the earliest known of the lines and points connected with the complete quadrilateral.

Then the pencils of rays with vertices at A_{14} and A_{13} respectively and passing through the points A_{34}, Y, Z_1, X are both harmonic. Hence A_{34}, Y, X are collinear.

Draw YT parallel to n''', cutting l_1 at T. Now triangles $TA_{13}Y$, $A_{12}Z_1D_{13}$ are similar, so that $TA_{13}\cdot D_{13}A_{12} = TY\cdot Z_1A_{12}$; and triangles $TA_{14}Y$, $A_{12}Z_1D_{23}$ are similar, so that $TA_{14}\cdot D_{23}A_{12} = TY\cdot Z_1A_{12}$. Hence $TA_{13}/TA_{14} = D_{23}A_{12}/D_{13}A_{12}$; and therefore $TA_{13}/A_{13}A_{14} = D_{23}A_{12}/D_{23}D_{13}$. Hence T is a fixed point; and therefore, if we consider the various possible inscribed conics, the locus of Y is a fixed line YT.

Now the pencil having T as vertex formed by rays passing through A_{34}, Y, Z_1, X is harmonic; therefore the locus of X is the fixed line XT. Let XT cut $A_{12}A_{34}$ in B_3. Then the points A_{34}, ∞, A_{12}, B_3 form a harmonic range; that is B_3 bisects $A_{12}A_{34}$.

Similarly we can show that the line which is the locus of centers of the inscribed conics bisects $A_{13}A_{24}$ at B_2 and $A_{14}A_{23}$ at B_1.*

Let U_1 be the center of gravity of the triangle $l_2l_3l_4$, U'_1 the center of gravity of equal masses placed at the three vertices that lie on the line l_1. Then

(9) The line U_1U_1' is concurrent with the three similar lines at their common mid-point U on the mid-diagonal line.†

The point U is the center of gravity of equal masses placed at (*a*) the six vertices; (*b*) the four centroids U_1, U_2, U_3, U_4; (*c*) the four points U_1', U_2', U_3', U_4'; (*d*) the three points B_1, B_2, B_3.

Place equal masses, m, at the six vertices. (*a*) Combine masses at opposite vertices; this gives three masses, $2m$, at B_1, B_2, B_3 whose center of gravity, U, is found by bisecting B_2B_3 at C and locating CU equal to one third of CB_1. (*b*) Replace three masses, m, at A_{34}, A_{24}, A_{23} by $3m$ at U_1, replace three masses, m, at A_{12}, A_{13}, A_{14} by $3m$ at U'. Then U bisects U_1U_1', and it is also the common mid-point of the three lines joining the mid-points of the opposite pairs of sides of the quadrangle $U_1U_2U_3U_4$.

(10) The lines joining $(A_{13}A_{14})$ to $(A_{23}A_{24})$ and $(A_{13}A_{23})$ to $(A_{14}A_{24})$ intersect on m.‡

Similarly two other points on m are located. The theorem follows at once from the fact that the diagonals of a parallelogram bisect each other.

4. The pedal line.

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* Simpler proofs that the middle points of the diagonals are collinear may be found in Casey's Sequel to Euclid, 1886, p. 5, in C. V. Durell's Plane Geometry for Advanced Students, Part I, 1909, pp. 118, 119, 188, and elsewhere.

Some remarkable members of the family of conics touching the four sides of the quadrilateral are discussed by Ripert, l. c.

† This theorem is given by Ripert, l. c.; it is proved, as above, by Neuberg, Annales de la Société scientifique de Bruxelles, 1902, p. 13.

‡ This theorem is given by Schlömilch, l. c.

(11) The feet of the perpendiculars from the focal point to the four sides of the quadrilateral are collinear.*

The line on which these four points lie will be referred to as the *pedal line* of the quadrilateral and it will be denoted by p.

Let G_1 be the foot of the perpendicular from F to l_1. The theorem follows at once from the Simson, Wallace or pedal line theorem.†

(12) At G_1 the circles drawn on FA_{12}, FA_{13}, FA_{14} as diameters concur.‡ Three circles similarly concur at G_2, G_3, G_4 on the pedal line.

If A, B, C, D are four concyclic points and H_a, H_b, H_c, H_d are the orthocenters of the triangles formed by omitting A, B, C, D respectively, it is easily seen that AH_a, BH_b, CH_c, DH_d bisect each other at a point N. So H_a, H_b, H_c, H_d is a group of four concyclic points on a circle whose radius is equal to that of the circle circumscribing $ABCD$.§ From the fact that the orthocenter of a triangle is the external center of similitude of the nine-point circle and the circumcircle of the triangle, and that the radii of these circles are in the ratio of 1 : 2, it follows that the nine-point circles of the four triangles ABC, ABD, ACD, BCD are concurrent at N. From the fact that the line joining a point on the circumcircle of a triangle to the orthocenter of the triangle is bisected by the pedal line of the first-named point, it follows that the pedal lines of D, A, B, C with respect to the triangles ABC, BCD, CDA, DAB respectively all pass through the point N.‖ Therefore, returning to the complete quadrilateral, we have

(13) The nine-point circle of the triangle $A_{23}A_{34}A_{24}$ is cut by the nine-point circles of $FA_{23}A_{34}$, $FA_{34}A_{24}$, $FA_{23}A_{24}$ at the point on the pedal line where FH_1 cuts it, H_1 being the orthocenter of the triangle $A_{23}A_{34}A_{24}$.

Three similar points exist.

Let F_1 be the point in $\mathcal{C}_1$ diametrically opposite to F. Draw F_1G_2', F_1G_3' perpendicular respectively to l_2, l_3; then $G_2'G_3'$ is the pedal line of F_1 with respect to the triangle $l_2l_3l_4$.

Now $G_3G_2A_{23} = G_3FA_{23} = (\pi/2) - FA_{23}A_{34}$; and

$$G_3'G_2'A_{23} = G_3'F_1A_{23} = (\pi/2) - A_{34}A_{23}F_1.$$

Therefore $FA_{23}A_{34} + A_{34}A_{23}F_1 = (\pi/2) = G_3'G_2'A_{23} + G_3G_2A_{23}$. Hence the pedal line of F_1 is perpendicular to the pedal line of the quadrilateral. Similarly the pedal lines of F_2, F_3, F_4 with respect to the triangles $l_3l_4l_1$, $l_4l_1l_2$, $l_1l_2l_3$ are perpendicular to the pedal line of the quadrilateral.

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* This theorem was given by Steiner, l. c., and proved by Davies, l. c.

† Mackay, l. c.; Mathematical Repository, old series, vol. 2, 1800, p. 111.

‡ Catalan, Théorèmes et problèmes, sixth edition, 1865, p. 34.

§ Mention, Nouvelles Annales, vol. 4 (1845), p. 654. Cf. (ϵ) and (μ) in part III.

‖ These facts are quoted by Alison, Proceedings of the Edinburgh Mathematical Society, vol. 3 (1885), pp. 79–93; they are referred by him to the Ladies' and Gentlemen's Diary for 1864, p. 55, and to the Reprints from the Educational Times, vol. 1, Question 1431.

Now let H_1F meet G_2G_3 at X_1, H_1F_1 meet $G_2'G_3'$ at X_1'. Let Y_1 be the middle point of C_1H_1. Then, since X_1 bisects FH_1, X_1' bisects F_1H_1, Y_1 bisects C_1H_1 and since C_1 is the center of the circle whose diameter is FF_1, Y_1 is the center of the circle whose diameter is X_1X_1'. But Y_1 is the center of the nine-point circle of the triangle $l_2l_3l_4$, and X_1, X_1' are on this circle. Hence G_2G_3 and $G_2'G_3'$ intersect at a point on the nine-point circle of $l_2l_3l_4$. Thus the pedal lines of the extremities of any diameter of the circumcircle of a triangle intersect at right angles on the nine-point circle of the triangle.* Hence

(14) The nine-point circle of the triangle $A_{23}A_{34}A_{24}$ is cut by the pedal line of F_1 with respect to that triangle at a second point on the quadrilateral's pedal line.†

Three similar points exist.

The following generalization of the pedal line may be mentioned:

(11*a*) If FG_1, FG_2, FG_3, FG_4 are drawn making any constant angle with l_1, l_2, l_3, l_4, the points G_1, G_2, G_3, G_4 are collinear.‡

3. **The orthocentric line.** From the well-known fact that the pedal line of a point with respect to a triangle bisects the join of that point to the orthocenter of the triangle,§ it follows that

(15) The orthocenters of the four triangles formed by the sides of the quadrilateral in threes are collinear in a line parallel to the pedal line and twice as far from the focal point as is the pedal line.‖

This line will be called the *orthocentric line*; it will be denoted by *o*.

(16) Circles drawn on the three diagonals of the quadrilateral as diameters are coaxal, their radical axis being the orthocentric line.¶

Let $A_{34}H_1$ cut l_2 at A_2, $A_{24}H_1$ cut l_3 at A_3, $A_{23}H_1$ cut l_4 at A_4. By similar triangles, $A_{34}H_1 \cdot H_1A_2 = A_{24}H_1 \cdot H_1A_3 = A_{23}H_1 \cdot H_1A_4$. But $A_{34}H_1 \cdot H_1A_2$ is equal to the square of the tangent from H_1 to the circle on $A_{34}A_{12}$ as diameter; and $A_{24}H_1 \cdot H_1A_3$ is equal to the square of the tangent from H_1 to the circle on $A_{24}A_{13}$ as diameter. These tangents may be imaginary. Then H_1 is a point on the radical axes of the three circles in pairs. Similarly H_2, H_3, H_4 are on these three radical axes. But, by (15), it follows that the three radical axes coincide with the orthocentric line.

As an immediate consequence we have the theorem

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* Casey, Sequel to Euclid, edition of 1886, Ex. 138, p. 164. Attributed to Graham in this, not in later, editions.

† In (13) and (14) I have made an obvious extension of theorems in the geometry of the triangle to the geometry of the quadrilateral.

‡ Poncelet, Propriétés Projectives, 1822, § 468.

§ E. g., Casey, Sequel to Euclid, p. 36.

‖ This theorem is given by Steiner, l. c., and is proved by Davies, l. c.

¶ This has been called Bodenmiller's theorem, for example by Schlömilch, l. c. It was stated without proof by Davies, l. c., in 1835; it is not in Steiner's list of 1828.

(16*a*) The mid-diagonal line is perpendicular to the orthocentric and pedal lines.*

Also

(17) The parabola π which touches the sides of the quadrilateral has F for its focus and the orthocentric line for its directrix.†

For the circle circumscribing the triangle formed by any three of these tangents passes through the focus; and the directrix passes through the orthocenter of that triangle.‡

(17*a*) The pedal line and the family of lines of (11*a*) all touch the parabola π.†

(17*b*) The parabola π also touches the six lines joining $(A_{13}A_{14})$ to $(A_{23}A_{24})$; $(A_{13}A_{23})$ to $(A_{14}A_{24})$; $\cdots$.§

This is easily proved by the method of polar reciprocation, as in part IV below, since the harmonically conjugate lines of L_1F with respect to the angle $L_3L_1L_4$ and of L_2F with respect to the angle $L_3L_2L_4$ are easily seen to concur at a point on the circumcircle, which is the reciprocal of the parabola.

(17*c*) The ends of chords parallel to l_1 from A_{34} in $\mathcal{C}_2$, from A_{24} in $\mathcal{C}_3$, from A_{23} in $\mathcal{C}_4$ are collinear with F in a line which cuts π at the point of tangency of the parabola with l_1.‖

Three other such lines exist.

For, if the line from A_{34} parallel to l_1, cuts $\mathcal{C}_2$ at U_2, join FU_2 and let it cut l_1 at T. Now, considering points on $\mathcal{C}_2$, $FU_2A_{34} = FA_{13}A_{34}$; and, considering that F, A_{13}, G_1, G_3 are concyclic, the latter is equal to FG_1G_3. Let the axis of π be drawn perpendicular to G_1G_3 and let it cut l_1 at S. Then $TSF = FG_1G_3 = FU_2A_{34}$. But, by alternate-interior angles, the latter is equal to FTS. Therefore, since $TSF = FTS$, T must be the point of contact of l_1 with the parabola π.

Similarly FU_3 and FU_4 cut l_1 at this point.

Draw $A_{24}L_3'$, $A_{34}L_2'$ perpendicular to l_1; draw $L_3'M_3'$, $L_2'M_2'$ perpendicular respectively to l_3, l_2. Let these lines intersect at V_1.

Now $A_{24}L_3'$ passes through H_3, $A_{34}L_2'$ through H_2. Let $L_3'V_1$, $A_{34}H_2$ cut at Y; $L_2'H_2$, $A_{14}H_3$ at Z. Then the anharmonic ratio of the pencil whose vertex is L_3' and whose rays pass through H_3, V_1, H_2, A_{12} is equal to the anharmonic ratio¶ of its transversal $(\infty\, YH_2L_2')$; but this is equal to H_2L_2'/H_2Y, which is equal to $A_{14}L_2'/A_{14}L_3'$.

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* This theorem is given by Steiner, l. c.

† This theorem is generally credited to Steiner, for example by S. Kantor, Sitzungsberichte der Akademie der Wissenschaften, Wien, vol. 76, part 2 (1878), p. 753. It is not however given in Steiner's note of 1828.

‡ Salmon, Conic Sections, edition of 1879, pp. 207, 275. C. Smith, Geometrical Conics, edition of 1899, pp. 48, 70.

§ This theorem is given by Ripert, l. c., and proved by him analytically.

‖ This theorem is new.

¶ The anharmonic ratio of four points on a line $(ABCD)$ is the value of $(AB\cdot CD/AD\cdot CB)$.

Again the anharmonic ratio of the pencil whose vertex is L_2' and whose rays pass through H_3, V_1, H_2, A_{12} is equal to the anharmonic ratio of its transversal ($H_3 \infty ZA_{14}$); but this is equal to $ZA_{14}/A_{14}H_3$, which is equal to $A_{14}L_2'/A_{14}L_3'$.

Therefore the pencils have equal anharmonic ratios and H_3, V_1, H_2 are collinear.

Similarly, if $A_{23}L_4'$ is perpendicular to l_1 and L_4M_4' is perpendicular to l_4, this last line passes through V_1.* V_1 is called the *orthopole* of l_1 with respect to the triangle $l_2l_3l_4$. Hence

(18) If perpendiculars are drawn from the vertices of one of the four triangles of the quadrilateral to the other side of the quadrilateral, and if then from the feet of these perpendiculars are drawn to the sides of the original triangle opposite the respective vertices, these three lines are concurrent at a point on the orthocentric line.

There are four such points.

(19) The center of the circle, $\mathfrak{D}$, circumscribing the diagonal triangle $D_{12}D_{13}D_{23}$ lies on the orthocentric line.†

For $(A_{13}D_{12}A_{24}D_{23})$ is a harmonic range; therefore any circle through D_{23} and D_{12} is orthogonal to the circle on $A_{13}A_{24}$ as diameter. Consequently the circle $\mathfrak{D}$ is orthogonal to the circles having $A_{12}A_{34}$, $A_{13}A_{24}$, $A_{14}A_{23}$ respectively as diameters. Hence, the theorem follows from (16).

Let $A_{23}H_1$ cut $A_{14}H_2$ in H_{12}' and $A_{14}H_3$ in H_{13}'; $A_{23}H_4$ cut $A_{14}H_2$ in H_{24}' and $A_{14}H_3$ in H_{34}'. Let $A_{23}H_1$ cut l_1 at S; let the orthocentric line cut l_1 at T_1. Now, since lines joining corresponding points are perpendicular to l_4, the anharmonic ratios of $(ST_1A_{13}A_{12})$ and $(H_1T_1H_2H_3)$ are equal; considering them as transversals across a pencil with vertex at A_{14}, we see that the anharmonic ratios of $(H_1T_1H_2H_3)$ and $(H_1SH_{12}'H_{13}')$ are equal; but the latter is the same as $(SH_1H_{13}'H_{12}')$. Hence

$$(ST_1A_{13}A_{12}) = (SH_1H_{13}'H_{12}');$$

and $A_{13}H_{13}'$, $A_{12}H_{12}'$ and the orthocentric line are concurrent. Thus it can be shown that

(20) The lines $A_{12}H_{12}'$, $A_{13}H_{13}'$, $A_{14}H_{14}'$, $A_{34}H_{34}'$ are concurrent at a point Z', on the orthocentric line.‡

Two other points are similarly found.

Let $A_{23}H_1$ cut l_4 at L_1^{iv} and $A_{23}H_4$ cut l_1 at L_4'. It is not hard to show that

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* According to Gallatly, this theorem is due to Neuberg. See his Modern Geometry of the Triangle, Chapter 6, where this and a number of other properties connected with the orthopole are worked out analytically. I have supplied the above proof and extended the theorem to the quadrilateral.

† Möbius, Berichte der Gesellschaft der Wissenschaften zu Leipzig, vol. 9 (1854), p. 87.

‡ This theorem is due to Terrier, Nouvelles Annales de Mathématiques, vol. 14 (1875), p. 514. I have supplied the proof.

(20*a*) The lines $L_1^{\text{iv}}L_4'$ and $L_2'''L_3''$ are concurrent at the point Z' of (20).[*]

Similarly a pair of lines passes through Z'' and a pair through Z'''.

It is obvious that

(21) The perpendiculars from $(A_{13}A_{14})$ to l_2 and from $(A_{23}A_{24})$ to l_1 are concurrent at (H_3H_4).[*]

There are six points of this kind on the orthocentric line.

(22) If lines are drawn from each vertex of the quadrilateral isogonally conjugate, with respect to the angle at that vertex, to the line joining that vertex to the focal point of the quadrilateral, these six lines are all perpendicular to the orthocentric line.[†]

For, F, A_{24}, G_2, G_4 are concyclic; hence, $FG_2G_4 = FA_{24}A_{34}$. Draw $A_{24}T$ isogonally conjugate to[‡] $A_{24}F$ with respect to $A_{12}A_{24}A_{34}$. Then $FG_2G_4 = A_{23}A_{24}T$. But FG_2 is perpendicular to l_2; therefore $A_{24}T$ is perpendicular to p; that is, $A_{24}T$ is perpendicular to o.

(23) Perpendiculars from F to l_3, l_1 cut $\mathcal{C}_1$, $\mathcal{C}_3$ respectively in points collinear with A_{24} in a line parallel to o.[§]

Five other such points exist.

For, $FG_1G_4 = FA_{14}A_{24} = FA_{12}A_{24} = FW_3A_{24}$, where W_3 is the point where FG_1 cuts $\mathcal{C}_3$, the first equation following from the fact that F, G_1, A_{14}, G_4 are concyclic. Therefore W_3A_{24} is parallel to p; similarly W_1A_{24} is parallel to p.

In the quadrilateral $A_{13}A_{14}A_{24}A_{23}$ denote the lengths of the sides in the order named by s_1, s_4, s_2, s_3. Let the perpendicular bisectors of s_1 and s_2 meet at T_{34}, and the perpendicular bisectors of s_3 and s_4 at T_{12}. Compare (6). Denote the distance of T_{12} from s_3 by t_{12}'''.

Now

$$T_{34}B_2^2 - T_{12}B_2^2 = \frac{T_{34}A_{24}^2 + T_{34}A_{13}^2 - T_{12}A_{24}^2 - T_{12}A_{13}^2}{2}$$

$$= \frac{s_2^2 + s_1^2 - s_3^2 - s_4^2}{8} + \frac{t_{34}''^2 + t_{34}'^2 - t_{12}^{\text{iv}2} - t_{12}'''^2}{2}.$$

Also

$$T_{34}B_1^2 - T_{12}B_1^2 = \frac{T_{34}A_{23}^2 + T_{34}A_{14}^2 - T_{12}A_{23}^2 - T_{12}A_{14}^2}{2}$$

$$= \frac{s_2^2 + s_1^2 - s_3^2 - s_4^2}{8} + \frac{t_{34}''^2 + t_{34}'^2 - t_{12}^{\text{iv}2} - t_{12}'''^2}{2}.$$

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[*] Terrier, l. c.

[†] Lachlan, Modern Pure Geometry, 1893, ex. 4, p. 68.

[‡] Two lines are *isogonally conjugate* with respect to an angle, if they pass through its vertex and make equal but oppositely directed angles with the sides of the angle.

[§] This is an extension of a theorem in Richardson and Ramsay, Modern Pure Geometry, 1893, ex. 15, p. 53.

Hence $T_{34}B_1^2 - T_{12}B_1^2 = T_{34}B_2^2 - T_{12}B_2^2$. Therefore $T_{12}T_{34}$ is perpendicular to B_1B_2. Hence

(24) The chords $T_{12}T_{34}$, $T_{13}T_{24}$, $T_{14}T_{23}$ are all parallel to the orthocentric line.*

Now T_{34} is diametrically opposite to A_{12} in the circle determined by $(A_{13}A_{14})$, $(A_{23}A_{24})$, A_{12}. But this circle goes through F, by (3). Hence FT_{34} is perpendicular to FA_{12}. Hence

(24*a*) If A_{12}''' is the point diametrically opposite to A_{12} in the circle $\mathcal{C}_3$, F, T_{34}, A_{12}''', A_{12}^{iv} are collinear.*

Five other sets of collinear points exist, all six lines being concurrent at F.

If T_{34}' is the point in the circumcentric circle diametrically opposite to T_{34}, then $T_{34}FT_{34}' = T_{34}FA_{12} = \pi/2$. Hence

(24*b*) The line $A_{12}F$ cuts the circumcentric circle at a point diametrically opposite to T_{34}.*

The following theorem is added, without proof:

(24*c*) The points F, A_{34}'', A_{24}''', A_{23}^{iv} are concyclic; and the center of this circle is diametrically opposite to C_1 on the circumcentric circle.*

There are three other such points.

The following interesting but isolated theorem may be added here without proof:

(25) The perpendicular bisectors of C_1H_1, C_2H_2, C_3H_3 and C_4H_4 are concurrent.†

6. **The bisectors of the angles.** Let the bisector of the angle through which l_1 must be rotated in the positive direction in order for it to coincide with l_2 be denoted by l_{12}. Let l_{14} and l_{23} intersect at K_{11}'; l_{14} and l_{32} at K_{12}'; l_{41} and l_{23} at K_{21}'; l_{41} and l_{32} at K_{22}'. l_{13} and l_{24} at K_{11}''; l_{13} and l_{42} at K_{12}''; l_{31} and l_{24} at K_{21}''; l_{31} and l_{42} at K_{22}''; l_{12} and l_{34} at K_{11}'''; l_{12} and l_{43} at K_{12}'''; l_{21} and l_{34} at K_{21}'''; l_{21} and l_{43} at K_{22}'''. Let the incenter of $l_2l_3l_4$ be I_1; the excenters opposite A_{34}, A_{24}, A_{23} respectively I_{12}, I_{13}, I_{14}. All these points are shown in Fig. 2.

(26) The lines

$$K_{11}'K_{11}''K_{22}''',\quad K_{11}'K_{22}''K_{11}''',\quad K_{22}'K_{11}''K_{11}''',\quad K_{22}'K_{22}''K_{22}'''$$

are the sides of a complete quadrilateral, $\mathcal{Q}_1$, and

$$K_{12}'K_{12}''K_{12}''',\quad K_{12}'K_{21}''K_{21}''',\quad K_{21}'K_{12}''K_{21}''',\quad K_{21}'K_{21}''K_{12}'''$$

are the sides of a complete quadrilateral, $\mathcal{Q}_2$.‡

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* This is given and proved analytically by Ripert, l. c.

† Lachlan, Modern Pure Geometry, 1893, Ex. 4, p. 92. Lachlan credits this theorem to Hervey, Educational Times Reprint, vol. 54. Another proof is given by Liénard in Mathesis, vol. 1 (1911), pp. 89–91.

‡ Mention, Nouvelles Annales, vol. 1 (1862), p. 76.

For the lines joining corresponding vertices of the triangles $I_{14}I_{13}I_{12}$, $I_{41}I_{32}I_2$ concur at I_1; hence, by Desargues' Theorem, the intersections of corresponding sides, i. e., K_{11}', K_{22}'', K_{11}''', are collinear. Similarly for the other sets of points.

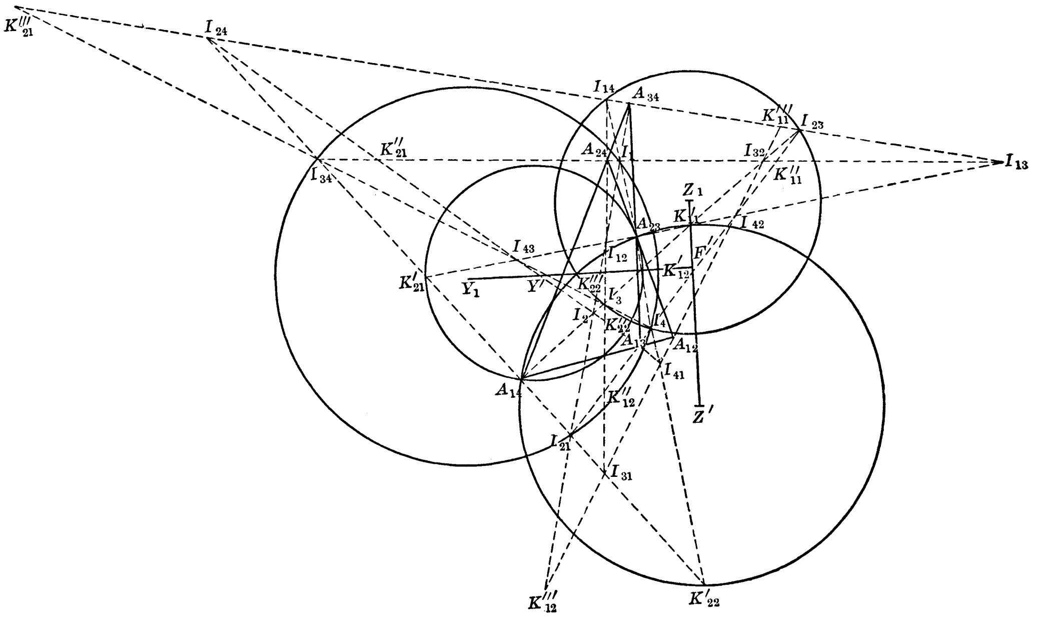

FIG. 2.

Now I_1, I_{21}, I_{34}, I_4 are concyclic. For

$$I_4I_{34}I_1 = \frac{A_{12}A_{24}A_{34} - A_{24}A_{12}A_{14}}{2};$$

and

$$I_4I_{21}I_1 = \frac{A_{12}A_{13}A_{34} - A_{14}A_{34}A_{13}}{2};$$

but $A_{12}A_{24}A_{34} + A_{14}A_{34}A_{13} = A_{24}A_{12}A_{14} + A_{12}A_{13}A_{34}$.

The diagonal triangle of the inscribed quadrangle $I_1I_4I_{21}I_{34}$ is $K_{22}'K_{11}''K_{22}'''$. Now the orthocenter of the diagonal triangle of an inscribed quadrangle is at the center of the circle.* But $K_{22}'K_{11}''K_{22}'''$ is one of the triangles of $\mathfrak{S}_1$. In this way it is proved that

(27) The sixteen centers of the circles inscribed and escribed to the four triangles of the quadrilateral are four by four concyclic, giving rise to eight new circles. These eight circles divide themselves into two groups,

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* Steiner, Crelle's Journal, vol. 30. Kantor, l. c. For $N'N''N'''$ is a self-conjugate triangle with respect to the circle.

such that the centers of four circles of one group are collinear (on s_1 the orthocentric line of the quadrilateral $\mathfrak{Q}_1$) and those of the other group are also collinear (on s_2 the orthocentric line of the quadrilateral $\mathfrak{Q}_2$).*

The circles are I_1, I_{21}, I_{34}, I_4, center Y_1; I_{12}, I_2, I_3, I_{43}, center Y_2; I_{13}, I_{23}, I_{32}, I_{42}, center Y_3; I_{14}, I_{24}, I_{31}, I_{41}, center Y_4. In the other group, I_{14}, I_{23}, I_3, I_4, center Z_1; I_{13}, I_{24}, I_{34}, I_{43}, center Z_2; I_{12}, I_{21}, I_{31}, I_{42}, center Z_3; I_1, I_2, I_{32}, I_{41}, center Z_4.

Now

$$Y_1I_1I_4 + I_4I_{14}Z_1 = (\pi/2) - I_4I_{21}I_1 + \pi/2 - I_{14}I_{23}I_4$$

$$= \pi - I_{23}I_{21}A_{34} - A_{34}I_{23}I_{21} = \pi/2.$$

Therefore $Y_1I_4Z_1 = \pi/2$. Hence z_1 is orthogonal to y_1. Hence

(27*a*) The circles of the first group, y, are respectively orthogonal to those of the second group, z; hence the lines s_1, s_2 are perpendicular.†

These lines will be called the *incentric lines* of the quadrilateral.

Now K_{22}' is the pole of $K_{11}''K_{22}'''$ with respect to y_1; therefore, since K_{11}', K_{11}'', K_{22}''' are collinear, K_{22}' and K_{11}' are conjugate points with respect to y_1. Hence the circle on $K_{22}'K_{11}'$ as diameter is orthogonal to y_1. Similarly circles on $K_{11}''K_{22}''$ and $K_{11}'''K_{22}'''$ as diameters are orthogonal to y_1, and these three circles are also orthogonal to y_2, y_3, y_4. In this way we find that

(27*b*) Three new circles may be added to each of the systems, y and z, of mutually orthogonal circles.‡

These circles are $K_{12}'A_{14}A_{23}K_{21}'$, center Y'; K_{12}'', A_{13}, A_{24}, K_{21}'', center Y''; K_{12}''', A_{12}, A_{34}, K_{21}''', center Y'''. In the other group, K_{11}', A_{14}, A_{23}, K_{22}', center Z'; K_{11}'', A_{13}, A_{24}, K_{22}'', center Z''; K_{11}''', A_{12}, A_{34}, K_{22}'', center Z'''.

It follows that

(28) The orthocentric line of the quadrilateral $\mathfrak{Q}_1$ is the mid-diagonal line of $\mathfrak{Q}_2$, and vice versa.†

Now K_{12}' is the orthocenter of the triangle $K_{11}'K_{21}'K_{22}'$. The nine-point circle of this triangle passes through A_{14} and A_{23}. It has the line joining $(K_{12}'K_{21}')$ to $(K_{11}'K_{22}')$ as diameter. But these are points, Y', Z', on s_1, s_2 respectively, which are perpendicular lines. Hence

(29) The intersection of s_1 and s_2 is on the nine-point circle of $K_{11}'K_{21}'K_{22}'$.†

Call this intersection F'. Then, considering points on the nine-point circle of $K_{11}'K_{21}'K_{22}'$, we have $A_{14}F'A_{23} = A_{14}Z'A_{23}$; but, since Z' is the center of the circle $A_{14}K_{11}'A_{23}$, the latter is equal to $2A_{14}K_{22}'A_{23}$ or $2(K_{22}'A_{14}A_{24} + A_{14}A_{24}A_{23} + A_{24}A_{23}K_{22}')$ or $A_{12}A_{14}A_{24} + A_{24}A_{23}A_{34} + 2A_{14}A_{24}A_{23}$ or $A_{13}A_{12}A_{23} + A_{14}A_{34}A_{13}$.

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* With the exception of the words in parentheses, this theorem is given by Steiner, l. c. The words in parentheses were added by Mention, l. c., who first proved the theorem. The above proof follows Mention closely, with a changed notation.

† Mention, l. c.

‡ Sancery, Nouvelles Annales, vol. 14 (1875), p. 145.

But, F being the focal point of the quadrilateral,

$$A_{14}FA_{23} = A_{13}A_{12}A_{23} + A_{14}A_{34}A_{13}.^*$$

For, $A_{13}FA_{23} = A_{13}A_{12}A_{23}$ and $A_{14}FA_{13} = A_{14}A_{34}A_{13}$.

Hence F' lies on the circle $A_{14}A_{23}F$; similarly on $A_{13}A_{24}F$ and on $A_{12}A_{34}F$. Since these three circles cannot be coaxal,† they have only the point F in common, so F' coincides with F. Hence

(30) The incentric lines intersect at the focal point of the quadrilateral.‡

The interesting fact seems not to have been noticed that

(31) The incentric lines bisect the angles $A_{14}FA_{23}$, $A_{13}FA_{24}$, $A_{12}FA_{34}$.

For, if $K_{12}'K_{21}'$ cuts $K_{11}'K_{22}'$ at W, then A_{14}, Y', A_{23}, F, W and Z' are all points on the nine-point circle of $K_{11}'K_{21}'K_{22}'$. Then

$$A_{14}FY' = A_{14}WY' = A_{14}WK_{21}'.$$

Also $Y'FA_{23} = Y'WA_{23} = K_{21}'WA_{23}$. But $A_{14}W$ and $A_{23}W$ are equally inclined to the altitude $K_{21}'W$. Therefore $A_{14}FY' = Y'FA_{23}$.

PART II. RELATIONS DERIVED BY AN INVERSION.

7. Inversion of a quadrilateral. It is well known that a complete quadrilateral, $\mathcal{Q}$, inverts with respect to its focal point as center of inversion into the four circumcircles of a second quadrilateral.§ This new quadrilateral, $\mathcal{Q}'$, is inversely similar to the original one,‖ the axis of similitude being one of the incentric lines; and points and lines connected with the new figure invert into points and lines connected in different ways with the original one. The incentric lines of the old are evidently also the incentric lines of the new quadrilateral. If we denote by $\mathcal{C}_j'$, the circle into which l_j inverts, and by l_j' the line into which $\mathcal{C}_j$ inverts, since the foot of the perpendicular from F on l_4' inverts into the point diametrically opposite to F on $\mathcal{C}_4$, clearly F and the four points diametrically opposite to F on the four circles $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_4$ are concyclic on a circle, the *diametral circle*, which touches the circumcentric circle of $\mathcal{Q}$ at F and has twice the radius. Hence, o', the orthocentric line of $\mathcal{Q}'$, being twice as far from F as p', by (15),

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[*] Davies, Mathematical Repository, vol. 6 (1835), Question 524.

[†] This is easily seen by inversion with F as center; compare part II.

[‡] This beautiful theorem is the last of Steiner's list; the proof is that of Mention, modified to fit Johnson's notation.

[§] McClelland, Geometry of the Circle, 1891, Ex. 12, p. 256; also Coolidge, Treatise on the Circle and Sphere, 1916, p. 87.

[‖] Clawson, American Mathematical Monthly, vol. 24 (1917), p. 71.

inverts into the circumcentric circle of $\mathcal{Q}$. Thus points on o' invert into points on $\mathcal{C}$, and points on $\mathcal{C}'$ invert into points on o. Throughout this part P' means the point inverse to P.

The following lemma will be needed:

(A) If the straight line AB be bisected at M, the inverse of M is the intersection of the circle $OA'B'$ with a circle of Apollonius of the triangle $OA'B'$ (the locus of a point P moving so that $A'P/PB' = A'O/OB'$). The perpendicular bisector of AB inverts into the circle of Apollonius named above.

I shall call M' the *Apollonian point* of A' and B' with respect to the point O.

8. **More lines through the focal point.** The following theorems are obtained by inversion from those indicated by the notation at the end of the theorems:

(32) A circle through F touching l_1 at A_{12} cuts the circle $\mathcal{C}_1$ at F and at a point on the diagonal n'''. There are twelve such circles determining four points on each diagonal (2).

(33) The Apollonian points of pairs of opposite vertices with respect to F determine a circle which passes through F (8).

(34) If a circle through F and A_{12} touching $\mathcal{C}_1$ cuts l_2 at λ_2'; a circle through F and A_{13} touching $\mathcal{C}_1$ cuts l_3 at λ_3'; a circle through F and A_{14} touching $\mathcal{C}_1$ cuts l_4 at λ_4'; then λ_2', λ_3', λ_4' are collinear with F in a line which cuts $\mathcal{C}_1$ at the point of tangency of $\mathcal{C}_1$ with a cardioide touching the four circumcircles, having F for its pole and with the diameter of the circumcentric circle produced its own length for its axis. There are three other such lines (17c).

(35) The point in which the diameter of $\mathcal{C}_1$ through F cuts l_3 and the point in which the diameter of $\mathcal{C}_3$ through F cuts l_1 are concyclic with F and A_{13} on a circle touching the circumcentric circle at F. Five other such circles exist (23).

9. **More points on the circumcentric circle.**

(36) Circles through F, A_{34} orthogonal to $\mathcal{C}_3$, through F, A_{14} orthogonal to $\mathcal{C}_1$, through F, A_{24} orthogonal to $\mathcal{C}_2$ are concurrent at a point on the circumcentric circle. There are three other such points (15).

(37) If the circle through F, A_{12} orthogonal to $\mathcal{C}_1$ cuts $\mathcal{C}_1$ in λ_2', the circle through F, A_{13} orthogonal to $\mathcal{C}_1$ cuts $\mathcal{C}_1$ in λ_3' and the circle through F, A_{14} orthogonal to $\mathcal{C}_1$ cuts $\mathcal{C}_1$ in λ_4', then the circles through F, λ_2' orthogonal to $\mathcal{C}_2$, through F, λ_3' orthogonal to $\mathcal{C}_3$ and through F, λ_4' orthogonal to $\mathcal{C}_4$ are concurrent at a point on the circumcentric circle. There are three other such points (18).

(38) The three circles through A_{14}, A_{23} orthogonal to the circle $FA_{14}A_{23}$,

through A_{13}, A_{24} orthogonal to the circle $FA_{13}A_{24}$ and through A_{12}, A_{34} orthogonal to the circle $FA_{12}A_{34}$ have two points in common which are both on the circumcentric circle (16).

(38*a*) The circle of (33) is orthogonal to the three circles of (38) and also to the circumcentric circle and to the diametral circle (16*a*).

(39) If circles through A_{14}, F orthogonal to $\mathcal{C}_1$ and through A_{23}, F orthogonal to $\mathcal{C}_2$ intersect at α_{12}'; if circles through A_{14}, F orthogonal to $\mathcal{C}_1$ and through A_{23}, F orthogonal to $\mathcal{C}_3$ intersect at α_{13}'; if circles through A_{14}, F orthogonal to $\mathcal{C}_4$ and through A_{23}, F orthogonal to $\mathcal{C}_2$ intersect at α_{24}'; and if circles through A_{14}, F orthogonal to $\mathcal{C}_4$ and through A_{23}, F orthogonal to $\mathcal{C}_3$ intersect at α_{34}'; then the circles $F\alpha_{12}'A_{12}$, $F\alpha_{13}'A_{13}$, $F\alpha_{24}'A_{24}$, $F\alpha_{34}'A_{34}$ are concurrent at a point ζ_1 on the circumcentric circle. There are three other such points (20).

(39*a*) If the circle through A_{14}, F orthogonal to $\mathcal{C}_1$ cuts $\mathcal{C}_1$ at λ_1^{iv}, the circle through A_{14}, F orthogonal to $\mathcal{C}_4$ cuts $\mathcal{C}_4$ at λ_4', the circle through A_{23}, F orthogonal to $\mathcal{C}_2$ cuts $\mathcal{C}_2$ at λ_2''' and the circle through A_{23}, F orthogonal to $\mathcal{C}_3$ cuts $\mathcal{C}_3$ at λ_3'', then the circles $F\lambda_1^{iv}\lambda_4'$ and $F\lambda_2'''\lambda_3''$ are also concurrent at ζ_1 (20*a*).

(40) Circles through F and the Apollonian point of A_{14} and A_{24} orthogonal to $\mathcal{C}_4$ and through F and the Apollonian point of A_{13} and A_{23} orthogonal to $\mathcal{C}_3$ are concurrent at a point on the circumcentric circle. There are five other such points (21).

10. More points on the orthocentric line.

(41) If o_1, o_2, o_3, o_4 denote the points of intersection of o with perpendiculars from F to the sides l_1, l_2, l_3, l_4 respectively, the circles of Apollonius through F of the three triangles $FA_{12}A_{13}$, $FA_{12}A_{14}$ and $FA_{13}A_{14}$ intersect at o_1 (4).

(42) If ϵ_1, ϵ_2, ϵ_3, ϵ_4 denote the points of intersection of o with l_1, l_2, l_3, l_4, the circles Fo_2A_{12}, Fo_3A_{13} and Fo_4A_{14} intersect at ϵ_1 (5).

(42*a*) Circles through F, A_{12} tangent to l_2, through F, A_{13} tangent to l_3 and through F, A_{14} tangent to l_4 meet at a point ϵ_1' on l_1, and

$$\epsilon_1 F \epsilon_1' = \pi/2 \ (5a).$$

(43) The circles of Apollonius through F of triangles $FA_{13}A_{14}$ and $FA_{23}A_{24}$ intersect at a point τ_{34} on o. There are five other such points (6).

(44) The line $F\tau_{34}$ is perpendicular to FA_{34} (24*a*).

11. The incentric lines. Two circles, passing through A_{34} and F, bisect the angles between $\mathcal{C}_1$ and $\mathcal{C}_2$. Call the one whose center, C_{12}, is so placed that FC_{12} bisects C_1FC_2, $\mathcal{C}_{12}$; call the one whose center, C_{21}, is so placed that FC_{21} bisects C_2FC_1, $\mathcal{C}_{21}$. Now $\mathcal{C}_{34}$, $\mathcal{C}_{23}$, $\mathcal{C}_{42}$; $\mathcal{C}_{34}$, $\mathcal{C}_{32}$, $\mathcal{C}_{24}$; $\mathcal{C}_{43}$, $\mathcal{C}_{23}$, $\mathcal{C}_{24}$; $\mathcal{C}_{43}$, $\mathcal{C}_{32}$, $\mathcal{C}_{42}$ are concurrent respectively meeting at ι_1, ι_{12}, ι_{13}, ι_{14}. In this way sixteen points are obtained.

(45) These points are concyclic in fours in such a way as to give two sets of four coaxal circles each. These sets are mutually orthogonal and their lines of centers are the incentric lines (27) and (27*a*).

(46) The points A_{14}, A_{23}, the intersection of $\mathcal{C}_{14}$, $\mathcal{C}_{23}$ and the intersection of $\mathcal{C}_{41}$, $\mathcal{C}_{32}$; A_{13}, A_{24}, the intersection of $\mathcal{C}_{13}$, $\mathcal{C}_{24}$, and the intersection of $\mathcal{C}_{31}$, $\mathcal{C}_{42}$; A_{12}, A_{34}, the intersection of $\mathcal{C}_{12}$, $\mathcal{C}_{34}$, and the intersection of $\mathcal{C}_{21}$, $\mathcal{C}_{43}$ determine three other circles belonging to one of these sets; also A_{14}, A_{23}, the intersection of $\mathcal{C}_{14}$, $\mathcal{C}_{32}$, and the intersection of $\mathcal{C}_{41}$, $\mathcal{C}_{23}$; and two other such sets of points determine three circles belonging to the other set (27*b*).

Again, if $\mathcal{C}_{14}$, $\mathcal{C}_{23}$ intersect at κ',

$$A_{14}\kappa' A_{23} = A_{14}\kappa' F + F\kappa' A_{23} = \tfrac{1}{2}A_{14}C_{23}F + \tfrac{1}{2}FC_{14}A_{23};$$

but

$$\begin{aligned}\tfrac{1}{2}A_{14}C_{23}F = (\pi/2) - C_{23}FA_{14} = (\pi/2) - (C_{23}FC_3 + C_3FA_{14})& \\ = -\tfrac{1}{2}C_2FC_3 + A_{14}A_{24}F&,\end{aligned}$$

on considering points on the circles $\mathcal{C}_{23}$, $\mathcal{C}_3$; and

$$\begin{aligned}\tfrac{1}{2}FC_{14}A_{23} = (\pi/2) - A_{23}FC_{14} = (\pi/2) - (A_{23}FC_1 + C_1FC_{14})& \\ = -\tfrac{1}{2}C_1FC_4 + FA_{24}A_{23}&,\end{aligned}$$

on considering points on the circles $\mathcal{C}_{14}$, $\mathcal{C}_1$; hence

$$\begin{aligned}A_{14}\kappa' A_{23} &= A_{14}A_{24}A_{23} + \tfrac{1}{2}(C_3C_1C_2 + C_4C_2C_1) \\ &= A_{14}A_{24}A_{23} + \tfrac{1}{2}(A_{24}FA_{34} + A_{13}FA_{34}) \\ &= A_{14}A_{24}A_{23} + \tfrac{1}{2}(A_{24}A_{23}A_{34} + A_{13}A_{14}A_{34}) = A_{13}A_{12}A_{23} + A_{14}A_{34}A_{13}.\end{aligned}$$

But it was shown, in part I, § 6, that $A_{14}K_{22}'A_{23}$ is equal to this same expression. Hence κ' lies in the same circle with A_{14}, K_{22}' and A_{23}; hence

(47) The circles of (46) are identical with those of (27*b*); and hence the circles of (45) are coaxal with the circles of (27).

12. **Summary.** There have now been enumerated sixteen straight lines passing through the focal point: four in (17*a*), six in (24*a*), four in (35) and the two incentric lines.

There are one hundred and thirty-four circles of more or less interest passing through the focal point: four circumcircles, one circumcentric circle, one diametral circle, three circles in (2), six in (3), six in (12), four in (24*c*), twelve in (32), one in (33), twelve in (34), six in (35), twelve in (36), twelve in (37), twelve in (39), six in (39*a*), twelve in (40), twelve in (41), and twelve in (42).

On the circumcentric circle forty-seven points have been mentioned: four circumcenters, one focal point, four points in (5), six in (6), three in

(7), six in (24*b*), four in (24*c*), four in (36), four in (37), two in (38), three in (39), and six in (40).

On the orthocentric line are thirty-eight points: four orthocenters, two points in (16), four in (18), one in (19), three in (20), six in (21), four in (41), four in (42), four in (42*a*), and six in (43).

On the mid-diagonal line there are seven points: three mid-diagonals, three in (10), one mean center, besides an indefinitely great number of centers of conics touching the sides of the quadrilateral.

On the pedal line there are twelve points: four in (11), four in (13), four in (14).

On each of the circles determined by F and a pair of opposite vertices there are eight points: four in (2), one in (7), the focal point, and a pair of vertices.

PART III. PROPERTIES OF A CYCLIC QUADRANGLE.

13. The mean center and the orthic center. It will be shown in part IV that the polar reciprocal of the complete quadrilateral with respect to a circle having F as center is a cyclic quadrangle, that is, a group of four points on a circle. From the properties of the cyclic quadrangle, $L_1L_2L_3L_4$, properties of the quadrilateral can be obtained by reciprocation. It therefore becomes desirable to collect in this part certain theorems concerning points and lines connected with the cyclic quadrangle.

(α) In any quadrangle the joins of the mid-points of the three pairs of opposite sides are concurrent.* The point of concurrency, E, is the center of gravity of four equal masses placed at the vertices (Fig. 3).

The point E is called the *mean center* of the quadrangle.

(β) The join of each vertex to the center of gravity of the triangle determined by the other three vertices passes through E.† The join of the intersection of the diagonals L_1L_3, L_2L_4 to the center of gravity of the quadrangle also passes through E.‡

For, equal masses m at each vertex may be replaced by $2m$ at (L_1L_2) and $2m$ at (L_3L_4) and these in turn by $4m$ at the bisector of the join of these points. Again the four equal masses may be replaced by $3m$ at G_1, the center of gravity of the triangle $L_2L_3L_4$, and m at L_1; and E must lie on G_1L_1 so that G_1E is one third of EL_1. Finally $m/3$ each at L_2, L_3, L_4 may be replaced by m at G_1; hence the four masses may be replaced by equal masses at G_1, G_2, G_3, G_4. But G, the center of gravity of the quadrangle, is at the intersection of G_1G_3 and G_2G_4. Since the quadrangles $\mathcal{L}$ and $\mathcal{G}$ are homothetic with E as center, the intersection of the diagonals N'' and the point G are collinear with E.

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* Catalan, Théorèmes et Problèmes, fourth edition, 1865, p. 8.

† Greiner, Archiv für Mathematik und Physik, vol. 60 (1877), p. 178.

‡ Deteuf, Nouvelles Annales, vol. 8 (1908), p. 442.

(γ) In a cyclic quadrangle perpendiculars from the mid-points of each side to the opposite sides are concurrent.*

We shall call this point the *orthic center* of the quadrangle, and denote it by H.

Denote (L_1L_2) by F_{12}. Let perpendiculars from F_{12}, F_{34} to L_3L_4, L_1L_2 respectively meet at H. Then $F_{12}H$ is parallel to $F_{34}O$, if O is the center of the circle, and $F_{12}O$ is parallel to $F_{34}H$. Hence O, E, H are collinear, and E bisects OH. Thus

(δ) The circumcenter, the mean center, and the orthic center are collinear, and $OE = EH$.†

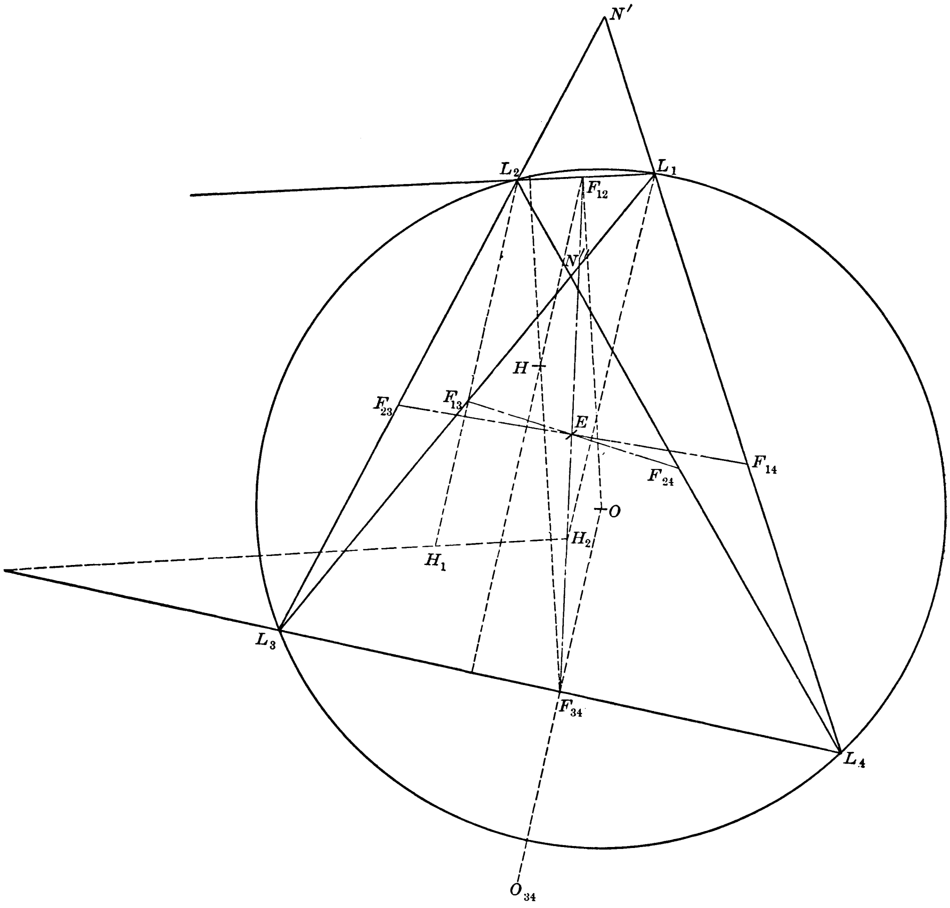

FIG. 3.

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* Greiner, l. c.; Kantor, l. c., quotes Pfaff as authority for this and some other theorems in connection with this point. Nager in Monatshefte für Mathematik und Physik, Wien, vol. 7 (1896), p. 325, refers it to Euler.

† Terrier, l. c. Also Greiner and Kantor.

Denote the orthocenter of the triangle $L_2L_3L_4$ by H_1. Now

$$L_2H_1 = 2OF_{34} = L_1H_2.^*$$

Hence L_1H_1 and L_2H_2 bisect each other at a point on a parallel to L_2H_1 through F_{12}, i. e., at H. Hence

(ϵ) The four lines joining each vertex to the orthocenter of the triangle determined by the other three vertices are concurrent at H.§

Again, the line joining L_1 to H_1 is bisected by the pedal line of L_1 with respect to the triangle $L_2L_3L_4$.† Hence

(ζ) The pedal line of each vertex with respect to the triangle determined by the other three vertices passes through H.‡

Further, if the image of O in L_1L_2 is O_{12}, since $F_{12}F_{34}$ is bisected at E, we have

(η) The line $O_{12}O_{34}$ is bisected at H.‖ There are three such lines.

Let L_1L_3 and L_2L_4 intersect at N''. Consider the triangle $N''F_{13}F_{24}$. Two of its altitudes intersect at H. Hence on considering the third altitude, we have

(θ) The perpendicular from the intersection of two opposite sides of the quadrangle to the line joining the mid-points of those sides passes through H.‖ There are three such lines.

Let the orthocenter of L_1L_2N'' be H_{12}''. Now $H_{12}''H_{23}''$ and $H_{14}''H_{34}''$ are each perpendicular to L_1L_3; also $H_{12}''H_{14}''$ and $H_{23}''H_{34}''$ are each perpendicular to L_2L_4. Hence $H_{12}''H_{34}''$ and $H_{14}''H_{23}''$ bisect each other at a point on a line through ($H_{14}''H_{34}''$) perpendicular to L_2L_4 and on a line through ($H_{12}''H_{14}''$) perpendicular to L_1L_3. But these are the lines of (γ). Hence

(ι) Six lines joining orthocenters of pairs of triangles formed by taking an intersection of two opposite sides of the quadrangle with two pairs of vertices pass through H.‖

However,

(ιa) There are only three distinct lines.¶

For, for example, H_{12}'', H_{34}'', H_{13}''', H_{24}''' are collinear on the orthocentric line of the complete quadrilateral L_1L_3, L_2L_4, $N''N'''$.

(κ) The radical axes of the three pairs of circles on pairs of opposite sides as diameters pass through H.¶

For, $HF_{12} = OF_{34} = R \cos L_3L_2L_4$, if R is the radius of the circumcircle, and $HF_{34} = OF_{12} = R \cos L_1L_3L_2$. Hence

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* Richardson and Ramsay, Modern Plane Geometry, p. 27.

† Casey, Sequel to Euclid, fourth edition, 1886, p. 36.

‡ Serret, Nouvelles Annales, vol. 7 (1848), p. 214.

§ Terrier, l. c.

‖ Deteuf, l. c.

¶ Kantor, l. c.

$$HF_{12}^2 - F_{12}L_2^2 = R^2(\cos^2 L_3L_2L_4 - \sin^2 L_1L_3L_2);$$

and $HF_{34}^2 - F_{34}L_3^2 = R^2(\cos^2 L_1L_3L_2 - \sin^2 L_3L_2L_4)$. But the right-hand sides of these equations are equal; hence $HF_{12}^2 - HF_{34}^2 = L_2F_{12}^2 - L_3F_{34}^2$.

Moreover, $N'L_2 \cdot N'L_3 = N'L_1 \cdot N'L_4$; $H_{12}''L_2 \cdot H_{12}''P_4'' = H_{12}''L_1 \cdot H_{12}''P_3'$; and $H_{34}''L_3 \cdot H_{34}''P_1''' = H_{34}''L_4 \cdot H_{34}''P_2^{\text{iv}}$, where P_4'' is the foot of the perpendicular from L_2 on L_1L_3. Hence N', H_{12}'', H_{34}'' lie on the radical axis of the circles with L_2L_3 and L_1L_4 as diameters.

(κa) The lines of (κ) and (ι) are identical.*

Evidently, from (ϵ), the quadrangle $H_1H_2H_3H_4$ is congruent with $L_1L_2L_3L_4$. It follows that

(λ) The join of the intersections of L_1L_2, H_3H_4; L_3L_4, H_1H_2 is bisected at H; similarly the joins of intersections of L_1L_3, H_2H_4; L_2L_4, H_1H_3 and the joins of intersections of L_1L_4, H_2H_3; L_2L_3, H_1H_4 are bisected at H.*

Kantor also proved that

(μ) All these intersections are collinear on the radical axis of the original circle and the circle circumscribing $H_1H_2H_3H_4$.*

This makes twenty-five straight lines meeting at H.†

Facts not so important for our present purpose are that the nine-point circles of the four triangles $L_2L_3L_4$,‡ $\cdots$; and of the four triangles $H_2H_3H_4$,§ $\cdots$ pass through H. The circle $N'N''N'''$ also goes through H.* Many other theorems related to this figure are given in the papers referred to in the foot notes, especially by Kantor.*

14. **The quadrilateral touching the circle at** $L_1L_2L_3L_4$. It follows at once from Pascal's theorem, considering the hexagon $L_1L_1L_3L_2L_2L_4$, that the intersection of tangents at L_1 and L_2 is collinear with N' and N''. Similarly considering $L_3L_3L_2L_4L_4L_1$, the intersection of tangents at L_3 and L_4 is collinear with N' and N''. Thus it appears that

(ν) The diagonal point triangle of the quadrangle $L_1L_2L_3L_4$ is the same as the diagonal triangle of the quadrilateral formed by the tangents to the circle at these points.‖

15. **Bisectors of the angles.** Denote the in-center and the three ex-centers of the triangle$L_2L_3L_4$ by I_1, I_{12}, I_{13}, I_{14}. Denote the mid-points of the arcs L_1L_2 by D_{12}, D_{21}, — D_{12} being on the bisector of $L_1L_3L_2$ (Fig. 4).

Now D_{12} is the center of a circle passing through L_1, I_3, I_4, L_2, I_{34}, I_{43}.

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* Kantor, Sitzungsberichte der Akademie, Wien, 1878, pp. 774–792; 1879, pp. 172–192; 757–763.

† Deteuf, l. c., mentions twenty-four such lines; he says twenty-seven, but see (ιa). Cikot, Nieuw Archief voor Wiskunde, Amsterdam, vol. 6 (1905), p. 63, mentions eighteen of them.

‡ Greiner, l. c.

§ Cikot, l. c.

‖ Catalan, Théorèmes et Problèmes, 1865, pp. 83, 95.

Hence $I_3I_4I_{34}I_{43}$ is a rectangle. Its sides I_3I_4, $I_{34}I_{43}$ are perpendicular to the bisector of $L_3D_{12}L_4$; that is, to $D_{12}D_{34}$. In fact the sides of the rectangle

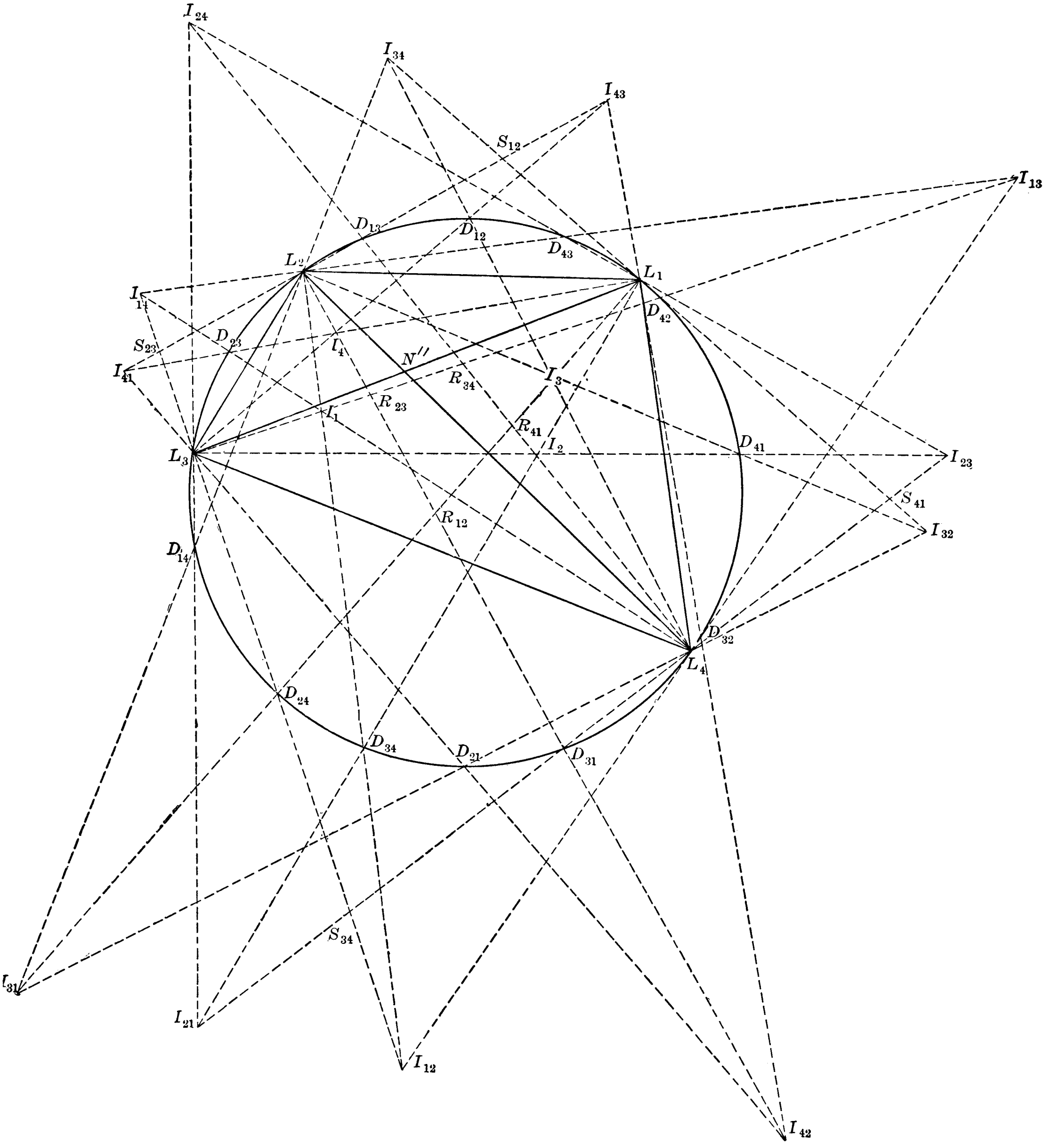

FIG. 4.

$I_3I_4I_{34}I_{43}$ are parallel to $D_{14}D_{32}$ and $D_{12}D_{34}$. But $D_{12}D_{34}$ is parallel to the bisector of $L_3N''L_4$; for, if $N''X$ bisects $L_3N''L_4$,

$$L_3N''X = \tfrac{1}{2}L_3N''L_4 = \tfrac{1}{2}L_3L_1L_4 + \tfrac{1}{2}L_1L_4L_2;$$

and, if $D_{12}D_{34}$ cuts L_1L_3 at Y,

$$L_3YD_{34} = L_3D_{12}D_{34} + L_1L_3D_{12} = \tfrac{1}{2}L_3L_1L_4 + \tfrac{1}{2}L_1L_3L_2.$$

Similarly $I_2I_3I_{23}I_{32}$, $I_1I_2I_{12}I_{21}$, $I_1I_4I_{14}I_{41}$, are rectangles whose sides are parallel to the same lines. Thus

(o) The centers of the sixteen in- and ex-circles are the intersections of four lines parallel to the bisector of the acute angle between the diagonals of the quadrangle and of four lines parallel to the bisector of the supplementary angle.*

Let L_1I_3 and L_2I_4 meet at R_{12}; let L_1I_{34} and L_2I_{43} meet at S_{12}. Now

$$L_2R_{12}L_1 = L_2L_1D_{24} + D_{31}L_2L_1 = \tfrac{1}{2}(L_2L_1L_4 + L_3L_2L_1);$$

and

$$L_3R_{34}L_4 = D_{42}L_3L_4 + L_3L_4D_{13} = \tfrac{1}{2}(L_2L_3L_4 + L_3L_4L_1).$$

But the right-hand sides are equal; hence

(π) The points R_{12}, R_{23}, R_{34}, R_{41} are concyclic. Similarly, S_{12}, S_{23}, S_{34}, S_{41} are concyclic.†

Now evidently $D_{24}D_{13}D_{42}D_{31}$ is a rectangle, for its diagonals are equal and bisect each other. But $D_{13}S_{12}D_{42} = D_{42}R_{34}D_{13} = \tfrac{1}{2}(L_2L_1L_4 + L_3L_2L_1)$; and

$$D_{13}D_{42}R_{34} = \frac{S_{12}D_{42}R_{34}}{2} = \tfrac{1}{2}L_1L_4L_3.$$

Hence the triangles $D_{42}S_{12}D_{13}$ and $D_{42}R_{34}D_{13}$ are congruent; and $S_{12}R_{34}$ is perpendicular to $D_{13}D_{42}$. But $R_{12}S_{12}L_1 = R_{12}L_2L_1 = \tfrac{1}{2}L_3L_2L_1$; and $D_{13}D_{24}L_1 = \tfrac{1}{2}L_3L_2L_1$. Hence $R_{12}S_{12}$ is parallel to $D_{13}D_{24}$. Hence S_{12}, R_{34}, R_{12} are collinear in a line parallel to $D_{13}D_{24}$. In fact

(ρ) The lines $D_{13}D_{24}$, $S_{12}R_{34}R_{12}S_{34}$, $D_{42}D_{31}$ are parallel; and $D_{24}D_{31}$, $S_{23}R_{23}R_{41}S_{41}$, $D_{13}D_{42}$ are parallel lines perpendicular to the first set.*

It is easily seen that $D_{13}D_{24}$ is parallel to $D_{12}D_{34}$; hence

(σ) The sets of parallel lines in (o) are parallel to those in (ρ).

If B_{12} bisects $S_{12}R_{12}$, $B_{12}L_2 = B_{12}L_1$. But $OL_2 = OL_1$. Hence OB_{12} is perpendicular to L_2L_1. Let the center of the circle circumscribing $R_{12}R_{23}R_{34}R_{41}$ be O_r, and the center of the circle circumscribing $S_{12}S_{23}S_{34}S_{41}$ be O_s. Then

$$\begin{aligned} S_{23}S_{12}O_s = (\pi/2) - S_{12}S_{41}S_{23} &= (\pi/2) - S_{12}D_{42}D_{13} \\ &= (\pi/2) - \tfrac{1}{2}L_1L_4L_3 = \tfrac{1}{2}L_1L_2L_3; \end{aligned}$$

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* This proof is given by Neuberg, Mathesis, vol. 6 (1906), p. 14. He refers to earlier statements of the theorem. For different proofs see American Mathematical Monthly, vol. 24 (1917), p. 429 and vol. 25 (1918), p. 241.

† Nager, l. c.

but $S_{12}L_2L_1 = \pi/2 - \frac{1}{2}L_1L_2L_3$. Hence $S_{12}O_s$ is perpendicular to L_2L_1. Similarly $R_{12}O_r$ is perpendicular to L_2L_1. Since similarly $S_{23}O_s$, $R_{23}O_r$ and $B_{23}O$ are perpendicular to L_2L_3, it follows that

(τ) The points O_r, O_s, O are collinear; and O bisects O_rO_s.*

Evidently by considering $L_1L_3L_2L_4$ as the inscribed quadrangle, we can find points R_{32}', R_{41}', R_{24}', R_{13}'; S_{32}', S_{41}', S_{24}', S_{13}'; and by considering $L_1L_2L_4L_3$ as the inscribed quadrangle we can find points R_{12}'', R_{43}'', R_{24}'', R_{13}''; S_{12}'', S_{43}'', S_{24}'', S_{13}'' for which similar statements can be made.

It follows from (o) and the consideration of the quadrangles of (τ) that

(υ) The bisectors of the angles N' and N''' are respectively parallel to those of N''.†

Thus we have two sets of sixteen parallel lines each, six chords like $D_{12}D_{34}$, four lines determined by in- and ex-centers, three lines like $S_{12}R_{34}R_{12}S_{34}$, and the three bisectors of the angles formed by opposite sides of the quadrangle.

PART IV. A POLAR RECIPROCATION.

16. Take any point, F, on the circumference of a circle, center O, passing through L_1, L_2, L_3, L_4. Draw any circle with F as center. The polar reciprocal of the first circle with respect to the second is the parabola, focus F, which touches l_1, l_2, l_3, l_4, the polars of L_1, L_2, L_3, L_4. Thus, by (17), we see that the cyclic quadrangle reciprocates into a complete quadrilateral, F being the quadrilateral's focal point. O, the center of the circle, evidently has for its reciprocal the orthocentric line, o, of the quadrilateral, the directrix of the parabola.

As noted in part III, properties of the cyclic quadrangle give rise to properties of the complete quadrilateral. The converse statement is true, but it is not our purpose further to develop that fact. We shall use the following lemmas:

(B) (L_1L_2) has for its reciprocal the straight line which forms a harmonic pencil with l_1, l_2 and the line joining the intersection of l_1 and l_2 with F. This line will be called the harmonic conjugate of $A_{12}F$.

(Ba) (L_1L_2) has for its reciprocal the symmedian line‡ through A_{12} of the triangle formed by l_1, l_2 and any line perpendicular to the line joining the feet of the perpendiculars from F on l_1, l_2.

If the triangle $L_1L_2L_3$ is reciprocated, a triangle $l_1l_2l_3$ results, whose vertices are A_{12}, A_{23}, A_{13}. This triangle is concyclic with F. Let the bisectors of the arcs $A_{12}A_{13}$ be U_{23}, U_{32}, $\cdots$ U_{23} lying on the bisector of $A_{12}A_{23}A_{13}$. Let FU_{23} cut $A_{12}A_{13}$ in V_{23}. Considering the Pascal hexagons $A_{23}U_{32}FU_{12}A_{12}A_{13}$ and $A_{23}U_{32}FU_{13}A_{13}A_{12}$, it follows that

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* Nager, l. c.

† This theorem is new.

‡ A symmedian line is a line isogonally conjugate to a median.

(*C*) The points V_{32}, V_{13}, V_{12} and I_4 are collinear, where I_4 is the incenter of the triangle $l_1l_2l_3$.*

It is easily seen that there are four of these "inial lines" belonging to the triangle $l_1l_2l_3$, one passing through the in- and one through each of the ex-centers of that triangle. V_{13}, V_{12}, V_{32} are the reciprocals of the bisectors of $L_1L_2L_3$, $L_2L_3L_1$, $L_3L_1L_2$ respectively; and the inial lines of $l_1l_2l_3$ are the reciprocals of the in- and ex-centers of the triangle $L_1L_2L_3$.

The following theorems are obtained by reciprocation from those indicated by the notation at the end of the theorems.

(48) The harmonically conjugate lines of $A_{12}F$, $A_{34}F$ with respect to l_1, l_2 and l_3, l_4 respectively intersect at a point which is collinear with the two similarly located points in a line e (α).

Otherwise, using (*Ba*),

(48*a*) The symmedian lines through A_{12}, A_{34} in the triangles l_1l_2m, l_3l_4m, where m is the mid-diagonal line, intersect at a point which is collinear with the two similarly located points in a line e (α).

(49) The intersection of l_2 with the harmonically conjugate line of $A_{34}F$ with respect to l_3, l_4, the intersection of l_3 with the harmonically conjugate line of $A_{24}F$ with respect to l_2, l_4 and the intersection of l_4 with the harmonically conjugate line of $A_{23}F$ with respect to l_2, l_3 are collinear on a line g_1, which intersects l_1 at a point on e. Three other such points on e are similarly found (β).

(50) The intersection of the line through F perpendicular to FA_{12} with the line through A_{34} harmonically conjugate to FA_{34} and the five other such intersections are collinear on a line, h (γ).

(51) The lines h and e intersect at a point on o (δ).

(52) If h cuts l_1 at H', H' also lies on the line h_1 determined by the intersection of l_4 with a perpendicular from F to FA_{23}, the intersection of l_2 with a perpendicular from F to FA_{34} and the intersection of l_3 with a perpendicular from F to FA_{24}. There are three other such points (ϵ).

(53) If a line through F perpendicular to FA_{12} cuts l_4 at M_3, a line through F perpendicular to FA_{23} cuts l_4 at M_1, and a line through F perpendicular to FA_{13} cuts l_4 at M_2, then $A_{12}M_3$, $A_{23}M_1$, $A_{13}M_2$ are concurrent at a point on h. There are three other such points (ζ).

(54) If a line from F perpendicular to FA_{12} cuts o at M_{12} and a line o_{12} is drawn so that o, $M_{12}A_{12}$, o_{12}, $M_{12}F$ form a harmonic pencil, and o_{34}, o_{13}, $\cdots$ are similarly obtained, then o_{12}, o_{34}; o_{13}, o_{24}; o_{14}, o_{23} intersect at three points on h (η).

(55) The perpendicular from F to the line joining F to the intersection

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* For a generalization of this theorem, see Clawson, American Mathematical Monthly, vol. 26 (1919), p. 59.

of the harmonically conjugate lines through A_{12} and A_{34} cuts n'' at a point on h. There are two other such points (θ).

(56) If the intersection of the perpendicular from F to FA_{23} with l_1 and the intersection of the perpendicular from F to FA_{14} with l_2 are joined, this line meets three other lines similarly drawn at a point on h. There are two other such points (ι).

(57) The line joining the intersection of h_1 and h_2 in (52) to A_{34}, the line joining the intersection of h_3 and h_4 to A_{12} and four other such lines all concur at a point on h (μ).

(58) The diagonal point triangle of the quadrangle determined by the points of contact of the parabola π with the four sides of the quadrilateral is the same as the diagonal triangle of the quadrilateral (ν).

(59) The sixteen inial lines (C) intersect by fours in four points which lie on a line through F; they also intersect by fours in four other points which lie on a line through F perpendicular to the first one. Call these lines s' and s'' (o).

(60) The line, b_{34}, joining the point where the bisector of $A_{23}FA_{24}$ cuts l_2 with the point where the bisector of $A_{13}FA_{14}$ cuts l_1, and the line, b_{21}, joining the point where the bisector of $A_{23}FA_{13}$ cuts l_3 with the point where the bisector of $A_{24}FA_{14}$ cuts l_4 intersect in a point on s'. There are six such points on s' and six on s'' (ρ).

(61) The line joining the point where the bisector of $A_{12}FA_{23}$ cuts l_2 and the point where the bisector of $A_{14}FA_{12}$ cuts l_1, the line joining the point where the bisector of $A_{34}FA_{23}$ cuts l_3 and the point where the bisector of $A_{14}FA_{34}$ cuts l_4 and two lines determined in the same way by the bisectors of the oppositely directed angles are concurrent at a point on s'. There are three such points on s' and three on s'' (σ).

(62) The points where the bisectors of $A_{12}FA_{34}$ and $A_{34}FA_{12}$ cut $A_{12}A_{34}$ are on s' and s'' respectively (υ).

Hence, from (31), it follows that

(63) The lines s' and s'' are identical with s_1 and s_2, the incentric lines.

Thus to the four points on each of the incentric lines mentioned by Steiner have been added three by Mention and Sancery (27*b*), four in this paper in (45), four in (59), six in (60), three in (61) and three in (62), making twenty-seven points in all on each incentric line, besides the focal point which is common to both.

URSINUS COLLEGE,
COLLEGEVILLE, PA., January 1, 1918.

Note.[*] The Editors of the Annals have just called to my attention a book, Premiers Éléments d'une Théorie du Quadrilatère Complet, by A.

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[*] This book was received for review by the Annals after the article of Mr. Clawson was in type, and a year and a third after it was submitted to the Annals for publication.—Ed.

Oppermann (Gauthier-Villars, 1919), which had just been received. Its author shows the same purpose exhibited in Ripert's paper and in mine—to assemble the important properties of the quadrilateral; like the present writer and unlike Ripert, M. Oppermann confines himself to the methods of pure geometry; but he covers a wider field than the writer in a less thorough way. There is some new material in the book, which covers sixty-two pages and is followed by a bibliographic supplement of ten pages more.

Most notable of the results not given in my paper are the last two theorems, XIX and XX. These theorems follow naturally after (16*a*) in part I above.

Oppermann proves that:

(16*b*) The circles of Monge* of the conics inscribed to the quadrilateral are coaxal with the circles having the diagonals as diameters (XX).†

He also proves that:

(16*c*) If three circles are drawn having for centers the intersections of the diagonals of the quadrilateral, and having for centers of similitude the six vertices of the quadrilateral (and this can be done in an infinite number of ways), the circle having for its center the radical center of the three circles and cutting them orthogonally, is coaxal with the four conjugate circles‡ of the quadrilateral (XIX).

The circles of (16*b*) and (16*c*) form two orthogonal systems of coaxal circles§—"*faisceaux conjugées de cercles*" (V).

Oppermann's theorem III is given by Ripert; but the proof is new:

(10*a*) Each of the triangles formed by three sides of the quadrilateral is crossed by the fourth side as a transversal. If on each side of the triangle a point is taken equally distant from the middle point of that side with the point where the transversal crosses that side, these three points lie on a second transversal of the triangle which is parallel to the mid-diagonal line. There are four such parallel lines (III).

The author gives my theorem (18) as an unpublished discovery of Goormaghtigh.

Besides this material, Oppermann gives in a different order about half of the theorems of part I of my paper, for the most part the classical theorems of Steiner's list and a few of Ripert's. Two of Ripert's theorems which he gives are related to theorem (9), but are not given above. Oppermann's proof of (30) is especially interesting and is somewhat shorter than that of Mention. The author does not always refer his theorems to their original sources, but his bibliographic supplement is reasonably complete, at least with respect to the French journals.

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* The *circle of Monge* of a conic is the locus of the intersection of perpendicular tangents to the conic. Kantor (see below) calls it the *Rechtkreis*. It is often called the *director circle*.

† This theorem will be found in Kantor, l. c., vol. 76 (1878), p. 774; also in Salmon, Conic Sections, Sixth Edition, 1879, p. 277.

‡ The *conjugate* (or *polar*) *circle* of a triangle is the circle with respect to which the triangle is self-conjugate (or self-polar). Its center is the orthocenter of the triangle, and it is real only for obtuse-angled triangles.

§ Kantor, l. c.

The book also contains an interesting classification of complete quadrilaterals, a test for similarity of quadrilaterals, a few properties of special types of quadrilaterals (a subject which has a fairly extensive literature of its own), and some other topics of minor interest.

The reader who wishes to pursue this subject further should not neglect Oppermann's book, nor the articles of Ripert, Terrier and Sancery cited above. Each of these papers extends the subject in its own special direction.

May 31, 1919.